Math and AI 4 Girls 2026 Problem Set Solutions

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Solutions

1. Every week, Alice tells the truth on exactly two days and lies on the other five days. She lies on the same days every week.

One week, Alice makes one statement daily on four consecutive days, in the given order:

Statement 1: If I tell the truth tomorrow, then today is Saturday.

Statement 2: I lie on Tuesdays and Saturdays.

Statement 3: Yesterday, I lied.

Statement 4: I will tell the truth exactly once in the next 3 days.

On which two days does Alice tell the truth, and on which days does Alice make each of the statements? Find all possible solutions.

Solution: We consider Statement 2, since doing so will restrict both Statement 1 and Statement 3.

Case 1: Statement 2 is true.

Then, Statement 3 is false. Additionally, Statement 1 must be false, since it can only be true if it is made on a Saturday, contradicting Statement 2. As a result, we have two subcases.

Suppose Statement 4 is false. Since we only have one true statement in the four given days, there must be exactly one in the other three. However, this means Statement 4 is true, which is a contradiction.

Suppose Statement 4 is true. Then, there will be three days where Alice tells the truth, which is not possible.

Therefore, Statement 2 cannot be true.

Case 2: Statement 2 is false.

Then, Statement 3 is true. Additionally, Statement 1 is vacuously true. Since this already gives us two true statements, all other statements must be false. This aligns with Statement 4 being false, so this assignment of truth values does not break.

Next, we must determine when each statement was made. In order for Statement 2 to be false, one of the truth days must fall on either Tuesday or Saturday.

Therefore, there are four solutions:

- Statement 1 is made on Tuesday, and Alice tells the truth on Tuesday and Thursday.
- Statement 1 is made on Thursday, and Alice tells the truth on Thursday and Saturday.
- Statement 1 is made on Saturday, and Alice tells the truth on Saturday and Monday.
- Statement 1 is made on Sunday, and Alice tells the truth on Sunday and Tuesday.

2. Let h be the cost of a hoodie, t be the cost of a t-shirt, and n be the cost of a notebook.

$$\begin{aligned} 2h + t + n &= 115 \\ h + 2t + 2n &= 110 \\ 2h + 2t &= 130 \end{aligned}$$

Solving the system of equations, we get that $h = 40$, $t = 25$, and $n = 10$, so the total price of a hoodie and notebook is $h + n = \boxed{50}$.

3. Let's say the number of people is n. Then, we have $n \equiv 5 \pmod 6$, $n \equiv 6 \pmod 7$, and $n \equiv 0 \pmod 5$. With the first two conditions, we can deduce

that n is 1 less than a multiple of $6 \cdot 7 = 42$, or $n \equiv 41 \pmod{42}$, because n is 1 less than a multiple of both 6 and 7. Trying values of n that satisfy this condition, we get that the minimum value is $n = \boxed{125}$.

4. Since MA_1A_{11} is equilateral, G lies on MA_{12}. Since the ratio of the lengths on the median that G splits the median into is $2:1$, we can calculate $MG = \frac{\sqrt{3}}{3}$. We know that $MA_4 = 1$, and $\angle A_4MG = 120°$, so the area of $\triangle MA_4G$ is

$$\frac{1}{2}(\sin 120°)(1)\left(\frac{\sqrt{3}}{3}\right) = \boxed{\frac{1}{4}}.$$

5. Let a, b, c be the roots of the polynomial

$$x^3 - 6x^2 + px - q = 0.$$

By Vieta's formulas,

$$a + b + c = 6, \qquad ab + bc + ca = p, \qquad abc = q.$$

We are given:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.$$

Using the identity

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab + bc + ca}{abc},$$

we obtain

$$\frac{p}{q} = 1 \implies p = q.$$

Next, we use the second condition:

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = 3.$$

Recall the identity

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 - 2\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right).$$

We already know:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1.$$

Also,

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{a+b+c}{abc}=\frac{6}{q}.$$

Substituting into the identity:

$$3=1^2-2\left(\frac{6}{q}\right)=1-\frac{12}{q}.$$

Solving for q:

$$\frac{12}{q}=-2 \implies q=-6.$$

$$\boxed{q=-6}$$

6. We notice that we can factor the fraction, giving us

$$\frac{(x+3)(x+4)(x+5)}{(x+6)(x+7)(x+8)}.$$

When we write out the product, we get

$$\frac{3\cdot4\cdot5}{6\cdot7\cdot8}\cdot\frac{4\cdot5\cdot6}{7\cdot8\cdot9}\cdot\frac{5\cdot6\cdot7}{8\cdot9\cdot10}\cdot\frac{6\cdot7\cdot8}{9\cdot10\cdot11}\cdots$$

We see the the $6\cdot7\cdot8$ in the denominator of the first fraction cancels out with the $6\cdot7\cdot8$ in the numerator in the fourth fraction. We note that

$$(x+3)(x+4)(x+5)=[(x-3)+6][(x-3)+7][(x-3)+8].$$

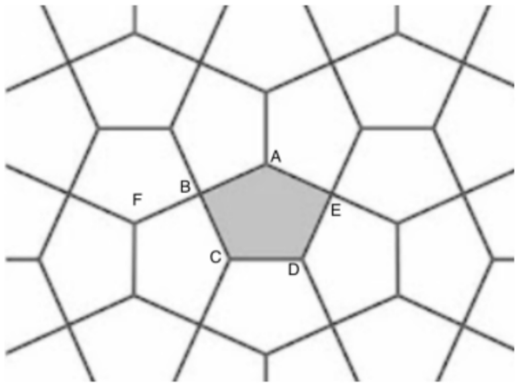
This shows that the product will telescope when we multiply it out.

Now we can multiply this out to get

$$\frac{3\cdot4\cdot5\cdot4\cdot5\cdot6\cdot5\cdot6\cdot7}{2029\cdot2030\cdot2031\cdot2030\cdot2031\cdot2032\cdot2031\cdot2032\cdot2033}$$

$$=\frac{3\cdot4^2\cdot5^3\cdot6^2\cdot7}{2029\cdot2030^2\cdot2031^3\cdot2032^2\cdot2033},$$

so $a+b+c+d+e+f+g+h+i+j=3+4+5+6+7+2029+2030+2031+2032+2033=\boxed{10180}$.



7. Label the graph like so.

Since

$$\angle ABC = \angle CBF, \quad \text{and} \quad \angle ABC + \angle CBF = 180^\circ,$$

$$\Rightarrow \angle AED = \angle ABC = 90^\circ.$$

Thus, $AC = AD = \sqrt{2}$. Let M be the midpoint of CD.

$$AM^2 = AC^2 - CM^2 = 2 - \frac{1}{4}.$$

$$\Rightarrow AM = \frac{\sqrt{7}}{2}.$$

So, the area of $ABCDE$ is

$$2 \cdot \left(\frac{1}{2} \cdot 1 \cdot 1\right) + \frac{1}{2} \cdot \frac{\sqrt{7}}{2} \cdot 1 = \boxed{\frac{\sqrt{7}}{4} + 1}.$$

8. **Solution 1**

The friends' monetary situation can be represented by the following generating function:

$$f(x) = (x^{-2} + x^{-1} + 1 + x + x^2)^4,$$

and we want to find the constant term in the expansion. When we expand the quartic, we get that the constant term is $\boxed{85}$.

Solution 2

Another way to solve this problem is by using a counting argument, breaking down the ways for the friends to collectively break even into casework.

Case 1: All four break even.

This can happen in one way.

Case 2: Three break even and one doesn't.

It is not possible for them to collectively break even in this case.

Case 3: Two break even and one friend loses money and one friend earns the amount lost.

There are $\binom{4}{2} = 6$ ways to choose the two to break even, 2 ways to choose whether the remaining two friends get $-1, 1$ or $-2, 2$, and 2 ways to choose which of those two friends wins. So there are $6 \cdot 2 \cdot 2 = 24$ ways for this case.

Case 4: One breaks even.

This can happen if one friend earns \$2 and the other remaining two both lose \$1, or vice versa. So this can happen in $4 \cdot 2 \cdot 3 = 24$ ways.

Case 5: None break even.

Two have to win and two have to lose. If they win an equal amount (i.e. both win \$2), the other two have to lose an equal amount. If they win different amounts, the other two also have to lose different amounts. Thus, there are

$$\binom{4}{2} \cdot (2 + 2 \cdot 2) = 36$$

ways.

So there is a total of $1 + 24 + 24 + 36 = \boxed{85}$ ways to break even collectively, matching the earlier answer.

9. This is a classic Van der Waerden type problem.

It is known that:

$$W(2, 3) = 9$$

Meaning that any 2-color coloring of $\{1, 2, \ldots, 9\}$ contains a monochromatic 3 term arithmetic progression. Since $\{1, 2, \ldots, 2026\}$ contains $\{1, 2, \ldots, 9\}$

as a subset, the same must be true. So, every 2 coloring of 1 through 2026 will contain a monochromatic 3 term arithmetic progression. Therefore, no valid colorings exist.

Final answer: $\boxed{0}$.

10. We note that

$$2026 = 2 \cdot 1013.$$

Thus, the only possible factorizations for the divisor count are

$$2026 = 2026 \quad \text{or} \quad 2026 = 1013 \cdot 2.$$

So, the exponents in the prime factorization of x must be either

$$2025$$

or

$$1012, 1.$$

To minimize x, we assign the largest exponent to the smallest prime. Thus,

$$x = 2^{1012} \cdot 3.$$

Note that

$$x = 2^{1012} \cdot 3$$

and

$$2^2 = 2^{22} \equiv 4 \pmod{100}.$$

So,

$$2^{1012} \cdot 3 \pmod{100} = 2^{12} \cdot 3 \pmod{100} = \boxed{88}.$$

11. Since AD is the diameter of the circle,

$$\angle ABD = \angle ACD = 90^\circ.$$

In right triangle $\triangle ABD$, we are given $AD = 10$ and $BD = 8$. By the Pythagorean Theorem,

$$AB^2 + 8^2 = 10^2.$$

We solve, finding $AB = 6$. Similarly, in right triangle $\triangle ACD$, we find $CD = 6$.

Since $ABCD$ is a cyclic trapezoid, it follows that $ABCD$ must be an isosceles trapezoid. Diagonals of an isosceles trapezoid are congruent and of the same length, thus, $AC = BD = 8$.

Since $ABCD$ is cyclic, Ptolemy's Theorem applies:

$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$

Substituting known values gives

$$(8)(8) = (6)(6) + (10)(BC),$$

and we solve, finding

$$BC = \frac{14}{5}.$$

Let h denote the height of trapezoid $ABCD$.

The area of $\triangle ABD$ can be computed in two ways. Using the legs,

$$[\triangle ABD] = \frac{1}{2}(AB)(BD) = \frac{1}{2}(6)(8) = 24.$$

And using base AD and height h,

$$[\triangle ABD] = \frac{1}{2}(AD)(h) = \frac{1}{2}(10)(h).$$

Equating the two expressions,

$$\frac{1}{2}(10)(h) = 24,$$

so we solve, getting

$$h = \frac{24}{5}.$$

Finally, using the area formula for a trapezoid,

$$[ABCD] = \frac{1}{2}(AD + BC) \cdot h = \frac{1}{2}\left(10 + \frac{14}{5}\right)\left(\frac{24}{5}\right) = \boxed{\frac{768}{25}}.$$

12. The prime factorization of 2025 is $3^4 \cdot 5^2$. Let $m = \gcd(n, 2025)$. In order for m to have 3 positive integer divisors, it must be of the form p^2, where p is a prime. For $n = 2025$, p must be 3 or 5.

First, we check when $m = 3^2 = 9$. Then the numbers that satisfy this are the multiples of 9 but not multiples of 5 or 27. The total number of multiples of 9 is $2025/9 = 225$. The number of multiples of 27 is $2025/27 = 75$. The number of multiples of 5 and 9 is $2025/45 = 45$. The number of multiples of 5 and 27 is $2025/135 = 15$. Thus, when $m = 9$, we have that there are

$$225 - 75 - 45 + 15 = 120$$

numbers.

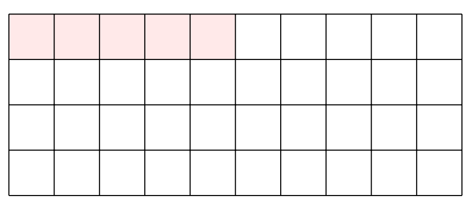
When $m = 5^2 = 25$, we want the numbers that are multiples of 25 but not multiples of 3. The number of multiples of 25 less than 2025 is $2025/25 = 81$. The number of multiples of 3 and 25 is $2025/75 = 27$. Thus, when $m = 25$, we have that there are

$$81 - 27 = 54$$

numbers.

Adding the two, we get that there are a total of $120 + 54 = \boxed{174}$ positive integers.

13. Consider the five columns that contain the five students in the first row (WLOG, the first five columns).



Then, one more student must be seated in those columns for a total of 5 students. There are 5 ways to partition those 5 students into the second, third, and fourth rows.

If the partition is $(5, 0, 0)$, there are 3 ways to permute the groups of students across the three rows. The rest of the seating is determined, so there are 3 seatings.

If the partition is $(4, 1, 0)$, there are 6 ways to permute the groups of students across the three rows, and $\binom{5}{1}$ ways to permute students within the rows. Similarly, there are $\binom{5}{1}$ ways to seat the rest of the students. So, the number of seatings is

$$6 \cdot \binom{5}{1}\binom{5}{1} = 150.$$

If the partition is $(3, 2, 0)$, the number of seatings is

$$6 \cdot \binom{5}{2}\binom{5}{2}\binom{5}{5} = 600.$$

If the partition is $(3, 1, 1)$, the number of seatings is

$$3 \cdot \binom{5}{3}\binom{2}{1}\binom{5}{4}\binom{4}{1} = 1200.$$

If the partition is $(2, 2, 1)$, the number of seatings is

$$3 \cdot \binom{5}{2}\binom{3}{2}\binom{5}{4}\binom{4}{2} = 2700.$$

This gives a total of

$$3 + 150 + 600 + 1200 + 2700 = 4653$$

seatings. Finally, there are $\binom{10}{5} = 252$ ways to choose the seats of the 5 students in the 1st row. Therefore, there are

$$4653 \cdot 252 = \boxed{1,172,556}$$

ways for the students to be seated.

14. We use Burnside's Lemma. We need to find the number of ways a bracelet is the same under its 11 transformations and then divide by 11.

Note that if the bracelet were to stay the same under any transformation (the identity and 10 other rotations) except the identity, all the beads would be the same color, which can only be blue in this case. So these 10 transformations each give 1 bracelet, for a total of 10.

For the identity, we need to find the number of ways she can arrange the beads so that no two green beads are adjacent. Let the number of ways to do this for $n+1$ beads be T_{n+1}. Let B_n be the number of ways to arrange $n+1$

beads given that the first bead is blue, and define G_n for green similarly (both with the constraint that no two green beads are adjacent). We have

$$T_{n+1} = B_n + G_n.$$

For B_n, if we put a blue bead next to it, we have B_{n-1} ways, and if we place a green, then we must place a blue next, giving us B_{n-2} ways. So

$$B_n = B_{n-1} + B_{n-2}.$$

For G_n, we must place two blues on both sides, leaving us with B_{n-2} ways.

Now, we need

$$T_{11} = B_{10} + G_{10} = B_{10} + B_8$$

from our recurrences. We have $B_1 = 2$ and $B_2 = 3$, and since B is the Fibonacci sequence shifted by one, we have $B_{10} = 144$ and $B_8 = 55$. So,

$$T_{11} = 199.$$

Finally, we apply Burnside's, and get

$$\frac{199 + 10}{11} = \boxed{19}$$

as the answer.

15. Let

$$R = \sqrt{A^2 + B^2},$$

and choose ϕ such that $A = R\cos\phi$ and $B = R\sin\phi$. With $x = \theta - \phi$, we get

$$A\sin\theta - B\cos\theta = R\sin x, \qquad A\cos\theta + B\sin\theta = R\cos x.$$

Hence

$$f(\theta) = \frac{(R\sin x)^3(R\cos x)^2}{R^5} = \sin^3 x\cos^2 x.$$

Let $a = \sin^2 x$ and $b = \cos^2 x$. Then $a, b \geq 0$ and $a + b = 1$, and

$$f(\theta)^2 = \sin^6 x\cos^4 x = a^3b^2.$$

Apply AM–GM to the 5 numbers $\frac{a}{3}, \frac{a}{3}, \frac{a}{3}, \frac{b}{2}, \frac{b}{2}$:

$$\frac{\frac{a}{3}+\frac{a}{3}+\frac{a}{3}+\frac{b}{2}+\frac{b}{2}}{5} = \frac{a+b}{5} = \frac{1}{5} \ge \left(\left(\frac{a}{3}\right)^3\left(\frac{b}{2}\right)^2\right)^{1/5}.$$

Raising to the 5th power gives

$$\frac{1}{5^5} \ge \left(\frac{a}{3}\right)^3\left(\frac{b}{2}\right)^2 = \frac{a^3b^2}{3^3\cdot 2^2},$$

so

$$a^3b^2 \le \frac{3^3\cdot 2^2}{5^5} = \frac{108}{3125}.$$

Therefore

$$f(\theta) \le \sqrt{\frac{108}{3125}} = \frac{6\sqrt{3}}{5^{5/2}} = \frac{6}{25}\sqrt{\frac{3}{5}}.$$

Equality occurs when $\frac{a}{3} = \frac{b}{2} = \frac{1}{5}$, i.e. $a = \frac{3}{5}$ and $b = \frac{2}{5}$, which is achievable (take x in the first quadrant).

Answer.

$$\boxed{\max_{\theta} f(\theta) = \frac{6\sqrt{3}}{5^{5/2}} = \frac{6}{25}\sqrt{\frac{3}{5}}.}$$

16. The following solution is not the pseudocode expected of the contestant, but a rough outline of the steps that can be followed in order to achieve the desired result.

Determine whether Die A wins against Die B (one way to do this is by comparing values with a nested loop and determining which die wins in more cases). Without loss of generality, assume that Die A wins against Die B. Then, in order for the three dice to be non-transitive, Die C must win against Die A and lose to Die B.

One way to determine if it is possible to construct such a Die C is to use six nested loops to test out every possible value of the numbers (each of the six loops will go from 1 to 8) on the face of Die C. For each of these six sets of values for the faces of Die C, test whether Die C wins or loses against Die A and wins or loses against Die B.

If Die C is able to form a set of non-transitive dice (i.e. Die C wins against Die A, Die C loses to Die B, and Die A wins against Die B), return "yes." Otherwise, if all the possible set of values for Die C have been tested and a valid set of values has not been found, return "no."