Math and AI 4 Girls Competition Problem Set Solutions

✅**Problem 1.** An animal shelter has a total of 20 cats, dogs, and parrots. There are at least twice as many cats as dogs and at least 3 times as many parrots as dogs. Find the maximum number of dogs they could have.

Methodology. Although you could solve this problem by guessing, the easiest and most efficient way to solve it is by using algebra. It is easy to get deterred by the "at least", but don't fret!
Solution. Suppose there are x dogs. Then there are at least $2x$ cats and $3x$ parrots. This means that there are at least $x+2x+3x=6x$ total animals. Since there are 20 animals, x is at most $20/6 = 3\frac{1}{3}$. Since there must be an integer number of dogs, we round **x** down to get **3**.

✅**Problem 2.** There exist three boxes of books: A, B, and C. Boxes A and B weigh a total of 10 pounds. Boxes B and C weigh a total of 9 pounds. Boxes C and A weigh a total of 11 pounds. Find the weight of each box.

Solution. Let a be the weight of box A, b be the weight of box B, and c be the weight of box C. Then, we can simplify the problem into the following system of equations:

$$a + b = 10 \quad (1)$$
$$b + c = 9 \quad (2)$$
$$c + a = 11 \quad (3)$$

There are a number of ways to go from here, but one particularly efficient method is to add the three equations to get:

$$2a + 2b + 2c = 10+9+11=30 \quad (4)$$
$$a+b+c=15 \quad (5)$$

Now we can subtract (1), (2), and (3) from (5), getting:

(a+b+c) - (a+b) = 15-10 = **c = 5**

a = 15-9 = 6

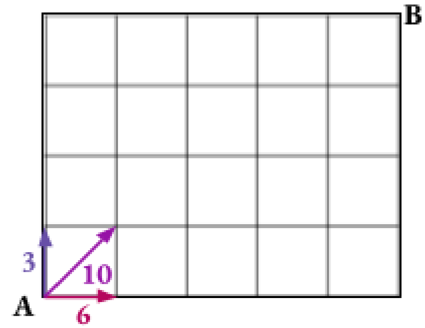
b = 15-11 = 4

Therefore, our answer is **a=6, b=4, c=5**.

✅**Problem 3.** An ant starts at point A and wants to get to point B. It can:

- Walk up a unit, which takes 3 seconds
- Walk to the right by one unit, which takes 6 seconds,
- Walk diagonally, which takes 10 seconds.

What is the minimum amount of time it can use to travel from point A to point B?



Solution. In an ideal solution, the ant should never walk diagonally. Let's prove this using a method known as "proof by contradiction."

> Suppose that the ideal solution involves a diagonal move. Now, we will prove that this isn't the ideal solution (there is some better solution). If we replaced that diagonal move with one upward move and one move to the right, a 10 second becomes two moves totaling 9 seconds; taking less time. Therefore, it is impossible for the ideal solution to involve any diagonal moves.

Any solution without diagonal moves involves 4 moves up and 5 moves to the right. Therefore, the minimum amount of time to travel from point A to point B is 3*4 + 6*5 = **42**.

✅**Problem 4.** Find nonzero digits A, B, and C that make the following equation true:

$$\begin{array}{r} 3A \\ \times B9 \\ \hline C08 \end{array}$$

Solution. Upon closer examination, we can see that the unit digit of the final product is **8.** Therefore, the unit digits of 3A and B9 must have a product ending in 8. The only digit A which satisfies this is A=2. Now, we have 32*B9= 288+320*B=C08. For the tens digit to be equal to 0, we must have B=1 or B=6. However, since C is a three-digit number, we must have **A=2, B=1, C=6.**

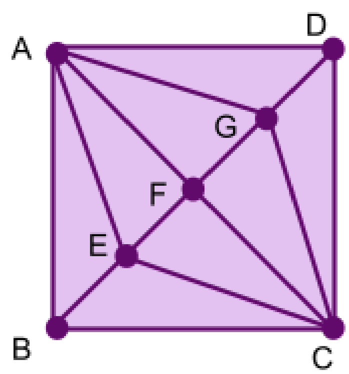
✅**Problem 5.** Every second, an elevator either has a 50% chance of going up and a 50% chance of going down. If the elevator starts at floor 15, what is the probability that, after 5 seconds, the elevator is at floor 18?

Solution. In order to go from floor 15 to floor 18 in 5 seconds, the elevator must go up 4 times and down 1 time. There are exactly 5 ways to do this, because the downwards move could be the 1st, 2nd, 3rd, 4th, or 5th move, and the rest of the moves are up. (Mathematically, this is expressed as 5! /(1! 4!)) Each move, there are 2 directions to move in, so there are a total of 2^5 = 32 ways to make 5 moves. Since 5 of these ways cause us to end on floor 18, our answer is **5/32**.

✅**Problem 6.** A three-digit number is called special if its digits form an increasing arithmetic sequence–that is, the difference between the first and second digit is the same as the difference between the second and third digit, and each digit is **larger** than the one to the left of it. For example, 567 is special, but 236, 111, 048, and 975 are not. How many three-digit special numbers exist?

Solution. We can split this problem into different cases by the common difference (the difference between the first and second digit, which is the same as the difference between the second and third digits). If the common difference is 1, our options are 123, 234, ..., 789. Therefore, there are 7 special numbers with common difference 1. If the common difference is 2, our options are 135, 246, ... 579. So there are 5 special numbers with common difference 2. If the common difference is 3, the options are 147, 258, 369, so 3 options. And finally, if the common difference is 4, only option is 159, so 1 option. The common difference can't be larger than 4. Therefore, there exist 7+5+3+1=16 three-digit special numbers.

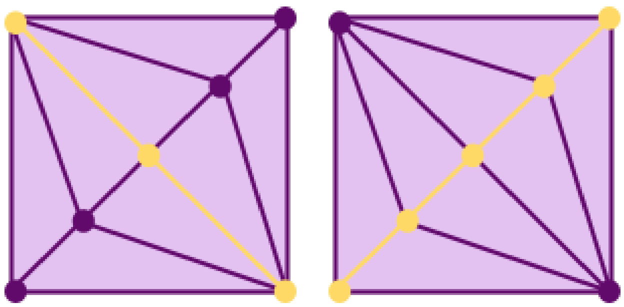
✅**Problem 7.** How many triangles are in the figure below?



Solution. This problem could be successfully solved by guess-and-check, but there is a way to make sure you have all the triangles. Consider line **BD**. There are 4 triangles that don't use any part of this line as a side (**AEC, ACB, AGC, ADC**) . All the other triangles must be either entirely above or entirely below this line. Let's count the number of triangles above. It must have the **A** as one of its points, and its other two points must be on **BD**. There are 5!/(3!2!) = 5*4/2 = 10 ways to choose the two points on the **BD**, so there are 10 triangles above it. Due to symmetry, there must also be 10 triangles below **BD**. Therefore, our answer is 4 + 10 + 10 = **24**.

Solution (Alternate). We can also solve this problem using complementary counting. To find the total amount of ways to choose three vertices from 7 points, we can do

$C_3^7 = 35$, but we are not done yet! There is a possibility that the three vertices we choose do not form a triangle, but a line. Thus, we can find the number of ways we can form a line with three vertices:



This gives us 35 - 1 - 10 = **24.**

✅**Problem 8.** A factory produces cars that are safe 80% of the time and uses a machine to test whether each car is safe. The machine marks safe cars incorrectly 10% of the time and unsafe cars incorrectly 40% of the time. If a car is marked as safe by the machine, what is the probability that it is actually unsafe?

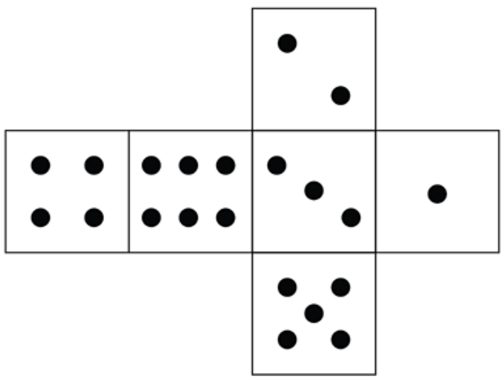
Solution. Assume that there are 100 cars. According to the problem, 80 of these cars will be safe and 20 will be unsafe. 10% of the 80 safe cars will be marked unsafe, meaning that 90% of these 80 cars, or 72 cars , will be marked correctly as safe. The 20 unsafe cars will be marked incorrectly as safe cars 40% of the time, meaning that 8 cars are incorrectly marked safe. In total, 72 +8 = 80 cars are marked as safe, however, 8 of those cars will actually be unsafe. Thus, the probability is simply 8/80, also expressed as **1/10** or **10%.**

✅**Problem 9.** A funky 6-sided die with faces labeled 1, 2, 3, 4, 5, and 6 has a $\frac{1}{12}$chance of landing on each face and a $\frac{1}{24}$ chance of landing on each edge. When Kira rolls the die, if it lands on a face, she records the number on top. If she lands on an edge, she records the sum of the two numbers on top. If Kira rolled the die infinitely many times and averaged the result, what is the expected value of the number recorded?

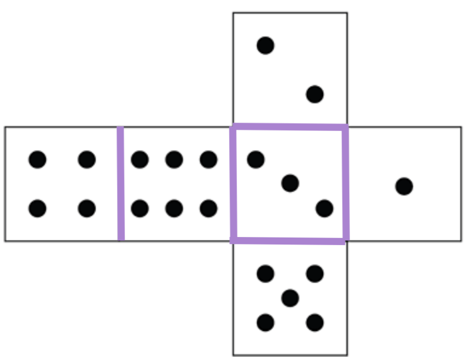
Solution. First, we can find the general outline to the solution of the problem. If we apply what we know about expected value into the problem, we get

$$Expected\ Value = \frac{1}{12}(Sum\ of\ each\ side) + \frac{1}{24}(Sum\ of\ adjacent\ sides)$$

From this outline, we know the sum of each side, which equals (1 + 2 + 3 + 4 + 5 + 6). Now, we just have to find the sum of the adjacent sides as well! Upon drawing a diagram of a dice, we can get:

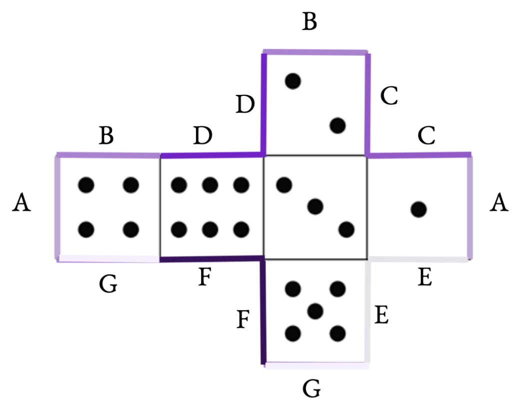


We already have the adjacent sides touching each other on the diagram already, which are simply:



(3,2), (3,6), (3,5), (3,1), (4,6).

But the rest of the pairs cannot be found as easily! We can label this layout with the sides that would touch each other, making the sides adjacent:



We can now see the values of the adjacent sides, leaving us with the pairs:
(1,5), (1,4), (2,4), (4,5), (1,2), (6,2), (5,6)
We can now plug in all the values, thus complete the problem.

$Expected\ Value = \frac{1}{12}(1 + 2 + 3 + 4 + 5 + 6) + \frac{1}{24}((3 + 2) + (3 + 6) + (3 + 5)$
$(3 + 1) + (1 + 5) + (1 + 4) + (2 + 4) + (4 + 5) + (1 + 2) + (6 + 2) +$
$(5 + 6) + (6 + 4))$
$= \frac{7}{4} + \frac{7}{2}$
$= \frac{21}{4}$
$= 5.25$

Hence, our answer is **5.25, or 21/4**.

Solution (Alternate). We still can use our initial outline; however, there is a simpler way to calculate the sum of the adjacent sides (if the dice lands on an edge). We know that there are 12 ways to pick two sides (as there are 12 ways to pick an edge). Since the average value of a side is 3.5, the average sum of two sides is 2(3.5)=7. Thus, our adjacent sides sum is 12(7)=**84**. Now, we have
$\frac{1}{12}(1 + 2 + 3 + 4 + 5 + 6) + 84/24 = 21/4.$

✅**Problem 10.**

The aliens that live on the distant planet of Zorblox use a different number base from us! In binary, or base 2, Ivy's favorite number is 11111100110. However, on Zorblox, her favorite number is stored as 5616. What is Ivy's favorite number in base 10? What number base do the Zorbloxians use?

Solution. First, let's convert 11111100110 into base 10. This is equal to 1(2)+1(4)+32+64+128+256+512+1024=**2022**. Neat, it's the current year! Now, we know that 5616 in the Zorbloxian base is equivalent to 2022. Upon observation, we can tell that the Zorbloxians use a smaller number than 10 as their base, since 5616 in base 10 would be significantly larger than 2022. Now, we know that their base is at least 7, otherwise, the digit '6' would not show up in their numbers. Therefore, they must use base 7, 8 or 9. Through the process of elimination, we see that the correct answer is **7.**

✅**Problem 11.** Suppose you have 4 teams: A, B, C, and D. Each team has a unique skill level, and in a game between any two teams, the team with the higher skill level always wins.

a. What is the **minimum** number of games needed to ensure that you know the teams' rankings? Write which teams compete in each 1v1 match.
b. Team E has arrived, and its players are eager to compete! Now, with 5 teams, what is the minimum number of games needed to ensure that you know their rankings?

Solution.

a) To determine the winner between A and B, we need one game. Without loss of generality, let's say A is better than B. Then, to place C into our ranking, we need at most 2 more games. If C loses against A in its first round, C would also need to compete against B before we determine C's placement. Without loss of generality, let's now say that the ranking is A, then B, then C. Now, to place D into the ranking, we can start with pitting D against B. If D loses, D competes against C. It D wins, it competes against A. Thus, we need 1+2+2=**5** matches to find a ranking.

b) Abridged solution:

Please note that there was an error in part b of the previous solution. The correct answer is that 7 games are necessary, not 8. The following is the correct solution to part b.

First, we will prove that 6 games are not enough to order all 5 teams. With 6 games, there are a maximum of 2*2*2*2*2*2 = 2^6 = 64 outcomes. However, there are 5*4*3*2*1 = 5! = 120 different ways that the 5 teams can be ordered. Since 120 > 64, a method with 6 games would not be able to produce every possible outcome, so it would not work.

Now we will show that it is actually possible to order the teams in 7 games. Without loss of generality, play A against B and C against D to get A>B and C>D. Without loss of generality again let A>C. Therefore, **A>C>D** and **A>B**. We have used 3 games so far. Now we will proceed with casework.

Let C play against E.

- **E>C:** play E, A.
    - **E>A**: **E>A>C>D** and **A>B**. Play B, C
        - **B>C**: E>A>B>C>D
        - **C>B**: play B, D
            - B>D: E>A>C>B>D
            - D>B: E>A>C>D>B
    - **A>E**: **A>E>C>D** and **A>B**. Play B, C
        - **B>C**: play B, E
            - B>E: A>B>E>C>D
            - E>B: A>E>B>C>D
        - **C>B**: play B, D
            - B>D: A>E>C>B>D
            - D>B: A>E>C>D>B
- **C>E**: play E, D
    - **E>D**: **A>C>E>D** and **A>B**. Play B, E
        - **B>E**: play B, C.
            - B>C: A>B>C>E>D
            - C>B: A>C>B>E>D
        - **E>B**: play B, D
            - B>D: A>C>E>B>D

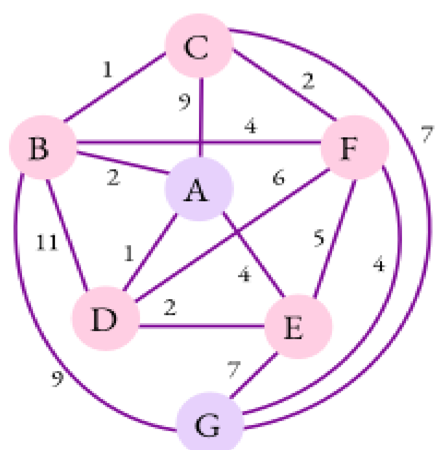
        - D>B: A>C>E>D>B
  - **D**>**E**: **A>C>D>E** and **A>B**. Play B, D
    - **B**>**D**: play B, C
      - B>C: A>B>C>D>E
      - C>B: A>C>B>D>E
    - **D**>**B**: play B, E
      - B>E: A>C>D>B>E
      - E>B: A>C>D>E>B

Therefore, the ranking can be determined in 7 games.

✅**Problem 12.** You are at city A, and your goal is to find the shortest route to city G! You may pass through cities B, C, D, E, and F. Each line connecting two cities has a number, which represents the number of days it takes to go between those two cities. Now, how many days (minimum) does it take to go from A to G, and what path do you take?[1]

Solution. Let's find the minimum distances between A and B, C, D, E, F. From there, we will add the travel time from each city directly to G.

1. The shortest path from A leads to D; it has length one. Therefore, it takes **one day** to go to city D. The shortest path from A to B is the direct one, which takes **two days**.
2. It takes 4 days to go directly from A to E. The paths from F to E is even longer, so we will not arrive at E from F. The only other way to arrive at E is from D. Since it takes 1 day to get to D and 2 to go from D to E, it takes **3 days** minimum to get from A to E.
3. Going directly from A to C is length 9. Finding the shortest route to C, we discover that it is through B, for a total of 2+1 = **3 days**.
4. We can reach F through B (2+4=6), C (3+2=5), D (1+6=7), and E (3+5=8). Here, the quickest is clearly through C, with a total of 5 days.
5. Finally, there are 4 cities connected to G. Traveling from B to G yields 2+9=11, E to G yields 3+7=10, F to G yields 5+4=9, and C to G yields 3+7=10. The shortest path is therefore **ABCFG, length 9**.



[1] Roads are not drawn to scale

Solution (Alternate). From A, we can go to B, C, D, and E.

- If we go to B first, we can then go to C, D, F, or G. Note that it is not optimal to go back to A. It is also not optimal to go to D, as it is faster to go from A straight to D, which will be covered in a later case.
    - If we go to C, we can then go to G or F, as we should not revisit A or B.
        - If we go to G, the path takes 10 days.
        - If we go to F, we can go to D, E, or G. However, again, we note that it is not optimal to go to D or E, as it is faster to visit them directly from A. Therefore, from F, we must go to G, which results in a path of 9 days.
    - If we go to F, it's not optimal to then go to C, as we already found a shorter path from A to C. Therefore, with the same logic as above, from F, the only reasonable step would be to go to G, which results in 10 days.
    - If we go to G, our path takes 11 days.
- Note that since A to C takes 9 days, if we go to C first, the journey must take at least 9 days. Since our minimum so far is 9 days, no path that starts from A to C would be optimal.
- If we go to D first, we can then go to E or F. Note that we should not go to B, as it would result in a journey of at least 12 days.
    - If we go to E, we can then go to F or G.
        - If we go to F, as shown above, we must then go to G, which results in a journey of 12 days.
        - If we go to G, our journey takes 10 days.
    - If we go to F, we must then go to G, for a total journey of 11 days.
- Note that since it's faster to go from A to D to E than to go straight from A to E, we do not need to consider the case where we go to E first from A.

Therefore, the shortest path takes **9** days and goes through towns **A, B, C, F, G**.

Solution (Alternate). We will use a table keeping track of the shortest paths to each town throughout the process.

	B	C	D	E	F	G
Shortest path	N/A	N/A	N/A	N/A	N/A	N/A

From A, we can go to B, C, D, and E, so we update the table accordingly.

	B	C	D	E	F	G
Shortest path	2	9	1	4	N/A	N/A

From this table, we can see that the shortest possible path to D is of length 1. Any other path must result in a length greater than 1, as it must first go from A to some town other than D. The shortest possible path to D is now fixed.

Now, we look at the towns we can visit from D and update the table accordingly.

	B	C	D-fixed	E	F	G
Shortest path	2	9	1	3	7	N/A

Of all the towns where the shortest possible path isn't fixed, B is the closest. This means that the shortest path to B must be 2, using the same logic as before.

Now, we look at the towns we can visit from B and update the table accordingly.

	B-fixed	C	D-fixed	E	F	G
Shortest path	2	3	1	3	6	11

We can repeat this algorithm of choosing the closest town from all the undetermined towns, fixing the shortest path from A to this town, and then updating the table by looking at the towns we can visit from this town.

Below show the steps of the algorithm for until G is fixed.

	B-fixed	C-fixed	D-fixed	E	F	G
Shortest path	2	3	1	3	5	10

	B-fixed	C-fixed	D-fixed	E-fixed	F	G
Shortest path	2	3	1	3	5	10

	B-fixed	C-fixed	D-fixed	E-fixed	F-fixed	G
Shortest path	2	3	1	3	5	9

	B-fixed	C-fixed	D-fixed	E-fixed	F-fixed	G-fixed
Shortest path	2	3	1	3	5	9

Therefore, the shortest possible path to G is of length **9**. We can then go back and look at when we updated the shortest journeys to each town to find the path **A-B-C-F-G**.