Joe is playing a game with infinite cups where he wants to maximize the number of points he earns. Joe plays by tossing ping pong balls into cups, starting with cup 1, and continues until he misses. He makes each shot with $\frac{1}{3}$ probability, and the number of points he earns is i, where the $(i+1)$th cup was his first miss. What is the expected number of points he earns? Express your answer as a common fraction.

Solution. The expected number of points he earns is the sum of the value of the cup times the probability that that cup is his last make. So, we have

$$E = \frac{1}{3} \cdot \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot 2 + \dots$$

To evaluate, we consider a new sum and subtract to cancel terms.

$$\frac{1}{3}E = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot 2 + \dots.$$

So,

$$E - \frac{1}{3}E = \frac{1}{3} \cdot \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} \cdot 1 + \dots,$$

$$\frac{2}{3}E = \frac{\frac{1}{3} \cdot \frac{2}{3} \cdot 1}{1 - \frac{1}{3}} = \frac{1}{3},$$

which means $E = \frac{1}{2}$.