MA4G 2023 Sample Solutions

MA4G Pset Team!

February 2023

1 Problems

Problem 1

Sophia wants to buy cookies that cost \$1.25 using as many coins as possible. She has dimes, nickels, and quarters with exactly 20 of each kind. What is the maximum number of coins Sophia can use to pay for her cookies?

We start by noting that the amount in cents for nickels, dimes, and quarters are 5, 10, and 25 respectively. By using 5 cents, we can use the maximum number of coins, as it has the least cost. We also note that 125 cents ends in 5, so there must either be an odd amount of 5's or 25s, but 25s are too large in value for there to be the maximum number of coins, therefore we use the maximum number of 5's which is 19(As there are 20 coins for the dimes, nickels, and quarters). This amounts to $19 \cdot 5 = 95$ cents. Because the total is 125 cents, we still need 30 cents worth of coins. The next lowest money value is dimes, so we know that 30 cents are worth $\frac{30}{10} = 3$ dimes. The total amount of coins is $3 + 19 = \boxed{22}$ coins.

Problem 2

Ivy organized a bake sale! She sold x ice cream bars, y pies, z cookies, and it satisfies the equation $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{5}{6}$. Find the triples of positive integers x, y, and z, where $x \leq y \leq z$.

Solution: Using the fact $x \leq y \leq z$ we get: $\frac{1}{x} + \frac{1}{x} + \frac{1}{x} \geq \frac{5}{6} > \frac{1}{x}$.From this, we find that the range of x is $2 \leq x \leq 3$. Because x is an integer, we can narrow down our search to 2 cases.

1. Case 1: When $x = 2$, we find the original equation to be $\frac{1}{2} + \frac{1}{y} + \frac{1}{z} = \frac{5}{6}$, and $\frac{1}{y} + \frac{1}{z} = \frac{1}{3}$. We can rearrange this equation to get $3z + 3y = yz$, and $(y-3)(z-3) = 9$. From this we find that $(y-3)$ and $(z-3)$ respectively can equal $(1, 9)$ or $(3, 3)$. Therefore in this case we get the triplets $(2, 4, 12)$ and $(2, 6, 6)$.

2. Case 2: When $x = 3$, we find the original equation to be $\frac{1}{3} + \frac{1}{y} + \frac{1}{z} = \frac{5}{6}$, and $\frac{1}{y} + \frac{1}{z} = \frac{1}{2}$. Rearranging, $(y-2)(z-2) = 4$. From this we find that $(y-2)$ and $(z-2)$ respectively can equal $(1, 4)$ or $(2, 2)$.

Thus, our final answer is $\boxed{(2,4,12),\ (2,6,6),\ (3,3,6),\ \text{and}\ (3,4,4).}$

Problem 3

Sam and Vaidehi have a coin tossing contest in which a fair coin is tossed 40 times. Sam will win if at least 21 consecutive heads show up, and Vaidehi will win if at least 21 consecutive tails show up. What's Sam's chance of winning the contest?

Solution: First, note that we can't have 21 consecutive heads and 21 consecutive tails at the same time. In a row of 40 coin tosses, let the start of the 21 consecutive tails be at position x. Then, the coin toss at position $x - 1$ (unless $x = 1$) must be heads, so 22 of the 40 coin tosses are determined. The remaining 18 coin tosses can be either heads or tails. If $x = 1$, then the 21 consecutive tails start at the start of the 40 coin tosses, so only 21 of the coin tosses are determined. There are 2^{19} ways for this to happen. Otherwise, when $x \neq 1$, we already found that for each x there are 2^{18} possibilities. x can range from 2 to 20, for a total of 19 possible values for x, so there are a total of $19 \cdot 2^{18}$ possibilities. Note that because the run of 22 coin tosses starts with heads, we are not over-counting anything.

The final probability is then $\frac{2^{19}+19\cdot 2^{18}}{2^{40}} = \frac{2^{18}\cdot 21}{2^{40}} = \boxed{\frac{21}{2^{22}}}$.

Problem 4

There are 200 girls who are participating in the Math and AI 4 Girls Competition. A survey was conducted and the results showed that 168 girls loved math, 199 enjoyed participating in competitions, 100 liked earth science, and 157 liked computer science. At least how many girls like all four?

Imagine that we give each participant a ticket for each activity she likes. Then, there will be a total of $168+199+100+157 = 624$ tickets given out. Now, imagine we take away a portion of those tickets. Participants who have 3 give back 3 tickets (the possible ticket left will represent the number of people involved in all 4 activities) and participants who have less than 3 tickets will give back all of their tickets. To minimize the number of students involved in all 4 activities, we want to maximize the number of students involved in at least 3 activities. Using this, we know that at most the number of tickets that can be taken back is $200 \cdot 3 = 600$. Finally, we calculate the number of tickets remaining: 624 - 600 = $\boxed{24}$.

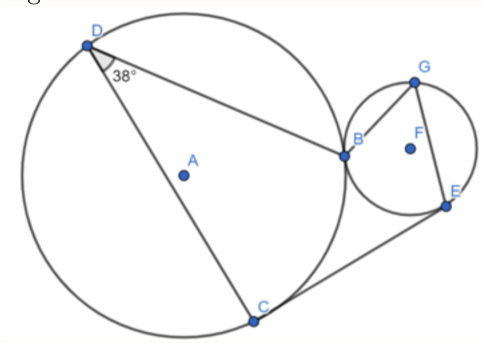
Problem 5

Angie and Anika are heading from point X to point Y, with speeds of 65 meters per minute and 87 meters per minute, respectively. At the same time, Natalie starts jogging from point Y and jogs toward point X with a speed of 112 meters per minute. The distance between point X and point Y is 1504 meters. How long does it take for Natalie to be exactly in the middle of Angie and Anika?

Let's imagine an invisible person, whose name is Chinmayi. Chinmayi is always exactly in the middle of Angie and Anika, so Chinmayi's speed is exactly the average of Angie's speed and Anika's speed, which is $\frac{(65+87)}{2} = \frac{152}{2} = 76$. We can see that when Chinmayi is in the middle of Angie and Anika, it is the same as when Natalie and Chinmayi meet. Since Natalie and Chinmayi are moving towards each other, we sum Chinmayi and Natalie's speed to find this rate to get 188. $\frac{1504 \text{ meters}}{188 \text{ meters per minute}} = \boxed{8 \text{ minutes}}$.

Problem 6

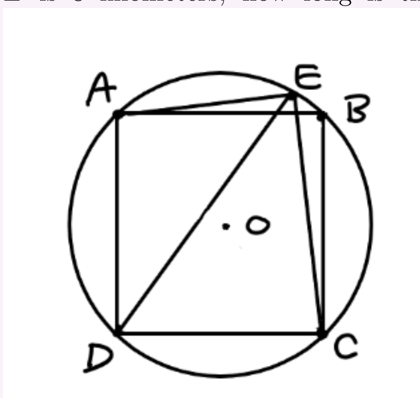
In the diagram as shown, $\angle CDB = 38°$, B, C, and E are points of tangency, and A and F are the centers of the two circles. What is the degree measure of $\angle BGE$? Note: The diagram is not drawn to scale.



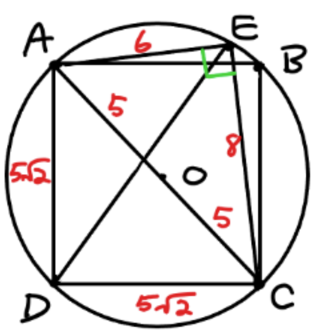
Draw lines AC, AB, FB, and FE from the center of the circles A and F. By the Inscribed Angle Theorem, $\angle BAC = 76°$ and $\angle ACE = \angle FEC = 90°$. Since A, B, and F are collinear, we can solve for $\angle BFE$. From quadrilateral $ACEF$, $\angle BFE = 360° - \angle ACE - \angle FEC - \angle FAC = 360° - 90° - 90° - 76° = 104°$. Using the Inscribed Angle Theorem again, $\angle BGE = \frac{1}{2}\angle BFE = \boxed{52°}$.

Problem 7

Towns A, B, C, D, and E, are positioned in a circle such that towns A, B, C, and D form a square. Town E is between towns A and B. Straight roads are built between the towns. If the road between towns A and E is 6 kilometers and the road between towns C and E is 8 kilometers, how long is the road between towns D and E?



Given that $ABCD$ is a square, AC must be a diameter of circle O. That means, arc ADC is $180°$, which means $\angle AEC = 90°$. Thus, AEC is a $6-8-10$ right triangle with $AC = 10$. Since AC is also the diagonal of square $ABCD$, the square has side length $\frac{10}{\sqrt{2}} = 5\sqrt{2}$. Finally, to find DE, we apply Ptolemy's Theorem: $AE \cdot CD + AD \cdot CE = AC \cdot DE \rightarrow 6(5\sqrt{2}) + 5\sqrt{2}(8) = 10(DE) \rightarrow DE = \frac{70\sqrt{2}}{10} = \boxed{7\sqrt{2}}$.



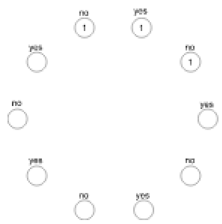
Problem 8

You have been placed in the center of a maze. Each room is indicated by a shape and paths are indicated by lines. On each move you make, you randomly pick a path to take, leading you to a different room, independent of any previous moves. Once you reach a star-shaped room, you win (and stop making new moves). What is the expected number of moves you will make before winning?

We can solve this problem using states. By symmetry, there are three possible states you could be in: 1) the circle. Let x be the expected number of moves it takes you to win from this room (we want to solve for x). 2) a square. Let y be the expected number of moves it takes you to win from this room. 3) a star. Let z be the expected number of moves it takes you to win from this room. Now, we must relate these states to each other. From the circle, each of the 4 possible paths lead you to a square. Thus, $x = \frac{4}{4}(y+1) = y+1$. Note that we must add the 1 because it took 1 move to get to that state. From each square, there is a $\frac{1}{3}$ chance to go back to the circle and a $\frac{2}{3}$ chance to get a star. Thus, $y = \frac{1}{3}(x+1) + \frac{2}{3}(z+1)$. Finally, once you reach a star you win, so $z = 0$. From our first equation, $y = x - 1$, so substituting that into our second equation gives $x - 1 = \frac{1}{3}(x+1) + \frac{2}{3}(0+1) \rightarrow \frac{2}{3}x = 2 \rightarrow x = 3$. Thus, you are expected to make $\boxed{3}$ moves before winning.

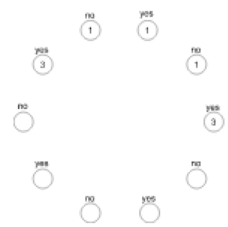
Problem 9

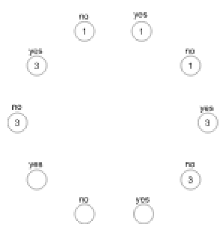
There are 10 people standing in a circle. Some of them always tell the truth. Some of them always lie. The rest of them sometimes tell the truth and sometimes lie. There is at least 1 of each type of person. When asked if they were standing next to at least 1 truth-teller, all of them responded yes. When asked if they were standing next to only truth-tellers, every other person responded with yes and the rest responded with no. Find all possible configurations. (Rotations and reflections do not count as different configurations).



Let 1 represent a truth teller, 2 represent a liar, and 3 represent everyone else. Notice that each truth-teller must stand next to another truth-teller, so there must be at least two truth-tellers standing next to each other. Exactly one of them must respond “yes” with the next question. This person must stand next to two truth-tellers.

Now there are two empty circles next to two truth-tellers. Consider either one of them. If this person is a liar, they would respond “no” to the first question, so this person can not be a liar. If this person is a truth-teller, the person next to them would’ve responded “yes” to the second question, so this person can not be a truth-teller.





Again consider either of the two blank circles next to a filled circle. This person can not be a liar. If this person is a truth-teller, the next two people must also be truth-tellers. However, then neither of the two remaining people can be liars. So, this person can not be a truth-teller.

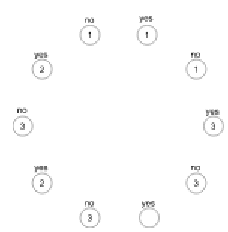
Now, we can do casework on where there is a liar. Notice that neither blank circle next to a filled circle can be a truth-teller.

Case 1: neither blank circle next to a filled circle is a liar.

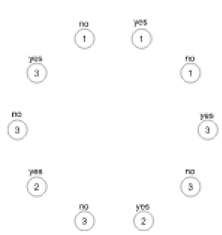
Then, the last person must be a liar. However, this person would respond “yes” instead of “no” to the second question, so this case does not work.

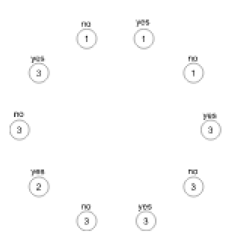
Case 2: at least one of these circles is a liar.

Then, the blank circle next to it can not contain a liar. It also can not contain a truth-teller because then the last circle must contain a truth-teller and we already know that the last circle can not contain a truth teller.



Now, with one circle left to fill, we can see that this can be either a liar or a person who sometimes lies and sometimes tells the truth.





Problem 10

Prove that $\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+...+\frac{1}{n^3}<\frac{3}{2}$ for every positive integer n, such that $n \geq 1$

We notice that there is no n term on the right side of the inequality, so we want to prove induction using a stronger inequality. We know that $\frac{3}{2}-\frac{1}{n}<\frac{2}{3}$, therefore, $\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+...+\frac{1}{n^3}<\frac{3}{2}-\frac{1}{n}<\frac{3}{2}$

Base Case:
We start by testing to see if 2 works as a base case, and it doesn't, so we next test 3 to see that it indeed does satisfy $\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}<\frac{3}{2}-\frac{1}{3}$

Inductive Step: We want to prove that: $\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+...+\frac{1}{n^3}+\frac{1}{(n+1)^3}<\frac{3}{2}-\frac{1}{n+1}$, and we are assuming that $\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+...+\frac{1}{n^3}<\frac{3}{2}-\frac{1}{n}$.
So we add $\frac{1}{(n+1)^3}$ to both sides of this equation to get:

$\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+...+\frac{1}{n^3}+\frac{1}{(n+1)^3}<\frac{3}{2}-\frac{1}{n}+\frac{1}{(n+1)^3}$.

Next, we know that $\frac{1}{(n+1)^3}<\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n-1}$.
Adding $\frac{-1}{n}$ to both sides, we see that $\frac{1}{(n+1)^3}-\frac{1}{n}<-\frac{1}{n-1}$.
Using this, we see that: $\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+...+\frac{1}{(n+1)^3}<\frac{3}{2}-\frac{1}{n}+\frac{1}{(n+1)^3}<\frac{3}{2}-\frac{1}{n+1}$.
Therefore, the conclusion follows from the principle of mathematical induction(PMI).
We have proved that $\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+...+\frac{1}{n^3}<\frac{3}{2}$ for every positive integer n, such that $n \geq 3$, so now we check for n = 1 and n = 2. We find that $\frac{1}{1^3}<\frac{3}{2}$ and $\frac{1}{1^3}+\frac{1}{2^3}<\frac{3}{2}$ are indeed true.

Problem 11

In an m by n grid of characters, where each character is either 1 or 0, we can draw a rectangle by letting 1s represent squares inside the rectangle and 0s represent squares outside the rectangle. In the diagram below, the rectangle is outlined in red.

0	0	0	0	0	0	0	0	0	0
0	1	1	1	1	1	1	0	0	0
0	1	1	1	1	1	1	0	0	0
0	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

However, sometimes, there isn't a perfect rectangle. Define the error to be the least number of squares that are labeled incorrectly across all possible rectangles in the grid.

Define an *almost-rectangle* to be a grid where the error is at most 4.

(a) Is the following an *almost-rectangle*?

0	0	0	0	0	0	0	0	0	0
0	0	1	1	1	0	1	0	0	0
0	0	1	1	0	1	0	0	0	0
0	1	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

(b) Given an m by n grid of characters where each character is either 1 or 0, write pseudocode that checks whether or not the grid contains an *almost-rectangle*.

a. $\boxed{Yes}$.

0	0	0	0	0	0	0	0	0	0
0	0	1	1	1	**0**	1	0	0	0
0	0	1	1	**0**	1	**0**	0	0	0
0	**1**	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

To find this solution, we notice that the 1s are all in rows 2 through 4 and columns 2 through 7, so we expect the rectangle to be in this area. Since there is only one 1 in column 2, it's more likely for the rectangle to not include column 2. This leads us to an answer.

b. Analysis: We can first count the total number of 1's in the grid. Then, given any rectangle, we can calculate the expected number of 1's if it was a perfect rectangle. We can also count the actual numbers of 1's in the rectangle. The difference between the actual number of 1's in the expected number of 1's in the rectangle is the number of squares that should be 1 but are actually 0. The difference between the total number of 1's and the number of 1's in the rectangle is the number of squares that should be 0 but are actually 1.

An example pseudocode is below.

/*read in input*/
Let **grid** be an empty grid
Set the dimensions of the grid to **m** by **n**
For each row from 1 to **m**:
    For each column from 1 to **n**:
        /*this describes a specific square in the grid*/
        Read in the value of the square
        Store the value in the **m**th row and **n**th column of **grid**

/*count total number of 1s*/
Let **total** be set to 0 and represent the total number of 1s

/*store the left-most, right-most, top-most, and bottom-most 1*/
Let **left** be set to **n**, **right** set to 1, **top** set to **m**, and **bottom** set to 1

For each row from 1 to **m**:
    For each column from 1 to **n**:
        /*this describes a specific square in the grid*/
        If this square contains 1:

Add 1 to **total**
Set **left** to the minimum of its current value and the current column number
Set **right** to the maximum of its current value and the current column number
Set **top** to the minimum of its current value and the current row number
Set **bottom** to the maximum of its current value and the current row number

Let **isAlmostRectangle** initially be equal to False

/*Iterate across all possible rectangles*/
For each **left_bound** from **left** to **right** (inclusive):
    For each **right_bound** from **left+1** to **right** (inclusive):
        For each **top_bound** from **top** to **bottom** (inclusive):
            For each **bottom_bound** from **top**+1 to **bottom** (inclusive):
                /*this describes a specific rectangle in the grid*/

                Let **expected** be set equal to
                    (**right_bound - left_bound**) * (**bottom_bound - top_bound**)

                Count the number of 1s in this rectangle using the same method we used to count the total number of 1s in the grid, and let this value be equal to **test**

                Let **error** be equal to (**expected - test**) + (**total - test**)
                If **error** is at most 4:
                    Set **isAlmostRectangle** to True
                    Exit all four loops immediately

The answer is the value of the variable **isAlmostRectangle**

Note: to improve the run time, when iterating across all possible rectangles, we can use 2d prefix sums instead to count the number of 1s.