MathandAI4Girls Solutions 2024

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1 Solutions

1. Using stars and bars, we see that we have 12 stars and 3 bars to separate the candy among 4 kids. Because everyone gets one candy, we can only place the bars in different spaces between the stars, giving us $\binom{11}{3} = 165$ ways to split the candy. Our second restriction is that nobody must get more than 6 pieces of candy. It isn't possible for more than 1 person to get over 6 pieces of candy because there isn't enough candy to give 2 people more than 6 pieces of candy and the rest at least 1 candy. Therefore, we only have to subtract the cases where one kid gets over 6 pieces of candy. To do this, we remove 6 candies from our original 12 and give them to one of the 4 kids. Then, we have 6 candies left to distribute to the 4 kids. In all of these cases, the kid who gets the 6 candies will get the first bucket, which will always have at least 1 candy and make them have more than 6 candies. The amount of ways to split the remaining 6 candies between the 4 kids is $\binom{5}{3} = 10$ (there are 5 spaces and 3 bars). For every kid getting over 6 pieces of candy, there are 10 ways to distribute the rest of the candy. Because we can give the 6 pieces of candy to any of the 4 kids, the total cases we are overcounting is $4 \cdot 10 = 40$. Because these cases are counted in our original count and we don't want to count them because they don't satisfy the restriction, there are a total of $165 - 40 = \boxed{125}$ ways to distribute the candy.

2. We rearrange the given equation like so:

$$(x + 5)(x + 8)(x + 6)(x + 7) = 5040.$$

   Expanding each group of two factors, we have

$$(x^2 + 13x + 40)(x^2 + 13x + 42) = 5040.$$

   This is the same as

$$(x^2 + 13x + 41 - 1)(x^2 + 13x + 41 + 1) = 5040,$$

   which can be rearranged using the difference of squares formula:

$$(x^2 + 13x + 41)^2 - 1 = 5040.$$

Adding one to both sides and taking the square root, we have two quadratic equations:

$$x^2 + 13x + 41 = 71 \text{and} x^2 + 13x + 41 = -71.$$

Solving the first equation, we have $\boxed{x = -15}$ and $\boxed{x = 2}$. The second equation has no real solutions since the discriminant is less than 0.

3. The shortest path should be a straight line from Tina's initial position to Tina's ending position upon "unraveling" the castle. The resulting shape is a rectangle with height 120 feet and width 100π feet (since the circumference of the circle becomes the width). We let Tina's initial position be the bottom-left corner of the rectangle, and her ending position must be the midpoint of the top side. Then, we are looking for the length of the hypotenuse of a right triangle with legs 120 feet and 50π feet which is $\boxed{\sqrt{120^2 + (50\pi)^2}}$ feet.

4. We can first write $abab_6$ and $bbc0_8$ in base 10 representations. By the definition of each base, we have $abab_6 = 6^3 \cdot a + 6^2 \cdot b + 6^1 \cdot a + 6^0 \cdot b = 216a + 36b + 6a + b = 222a + 37b$ and $bbc0_8 = 8^3 \cdot b + 8^2 \cdot b + 8^1 \cdot c + 8^0 \cdot 0 = 512b + 64b + 8c + 0 = 576b + 8c$. We can now se these two expressions equal to each other to get $222a + 37b = 576b + 8c$. We are also given that $c = 2b$, so substituting this in for c, we get $222a + 37b = 576b + 8(2b) = 576b + 16b = 592b$. Simplifying, we have $222a + 37b = 592b \rightarrow 222a = 555b \rightarrow 2a = 5b$. The only integers that satisfy this equation while remaining valid numbers to use in base 6 and 8 are $a = 5$ and $b = 2$. Now, we simply must compute $abab_6$. We know this is equal to $592b$, so $592(2) = \boxed{1184}$.

5. Let r be the rate at which she started painting her room, and t be the amount of time she spent, in minutes, from the start to the end of the break. Then, she spent t-15 minutes before the break painting and t minutes after the break painting. We get the equation

$$\begin{aligned} r(t - 15) &= 3/4rt \\ rt - 15r &= 3/4rt \\ 1/4rt &= 15r \\ t &= 60 \end{aligned}$$

The total time spent is 2t, so she spent a total of $\boxed{120}$ minutes.

6. We know that $\angle AFB \cong \angle CFE$ because of Vertical Angles. We also know $\angle FAB \cong \angle FCE$ and $\angle FBA \cong \angle FEC$ because of Alternate Interior Angles. By AAA similarity, $\triangle AFB \sim \triangle CFE$. The ratio of the sides is given

as 3 : 2. This means the height of $\triangle AFB$ from AB is $\frac{3}{2}$ of the height of $\triangle CFE$ from CE. This means the height of $\triangle AFB$ from AB is $\frac{3}{5}$ of the total height of the square. If we draw a line GH parallel to AB passing through F where G is on BC and H is on AD, this splits the square into two rectangles where rectangle $ABGH$ has area $\frac{3}{5}$ of the total square. We can see that the area of $ABGH$ is twice the area of $\triangle AFB$. Therefore, $[ABGH] = 54$ and the square's area is $54 \cdot \frac{5}{3} = \boxed{90}$.

7. We look at the use of consecutive integers and see that for some integers x and y, $x + (x+1) + (x+2) + ... + (x+y-1) = 504$. This sum can be rewritten as: $(2x+y-1) * y = 2 * 504 = 2^4 * 3^2 * 7$. We know that $y < 2x+y-1$ and that the 2 terms being multiplied have different parity, and we also want the value for y to be such that it is close to 2x + y - 1. This gives $y(2x+y-1) = 21 * 48$. From this, we can see that y = 21 and x = 14. Therefore, the greatest number of people who can buy her dance costumes is $\boxed{21}$ people.

8. Notice that $\frac{2x^3-7x^2-2x+28}{2x-7} = \frac{2x^3-7x^2-2x+7+21}{2x-7} = \frac{(2x-7)(x^2)+(2x-7)(-1)+21}{2x-7} = \frac{(2x-7)(x^2-1)+21}{2x-7} = \frac{(2x-7)(x^2-1)}{2x-7} + \frac{21}{2x-7} = (x^2-1) + \frac{21}{2x-7}$. Clearly x^2-1 is an integer since x is a positive integer. Then, we must have $\frac{21}{2x-7}$ be an integer which occurs when $2x-7$ divides $21 = 3*7$. That means $2x-7 = 21, 7, 3$, or 1, so $x = 14, 7, 5$, or 4. Thus, our answer is $14*7*5*4 = \boxed{1960}$.

9. Suppose the king and queen are in the same row. Each row consists of 8 squares, so there are $\binom{8}{2}$ ways to choose 2 squares to place the pieces on. Additionally, the two pieces are distinct, so we must multiply by 2!. Thus, there are $2! \cdot \binom{8}{2} = 56$ possible placements in a given row. We then multiply by the number of rows to get $56 \cdot 8 = 448$ total ways the king and queen can be in the same row. By symmetry, there are also 448 ways the king and queen can be in the same column.

   Now suppose the king and queen are in the same diagonal. We define the "length" of a diagonal as the number of squares it contains. A diagonal of length 1 does not satisfy the conditions as the two pieces must be placed on distinct squares. So, we consider diagonals of lengths 2 through 8. There are exactly 4 of each of the diagonals of lengths 2 through 7, and exactly 2 diagonals of length 8. Therefore, the number of cases in which the king and queen are in the same diagonal is

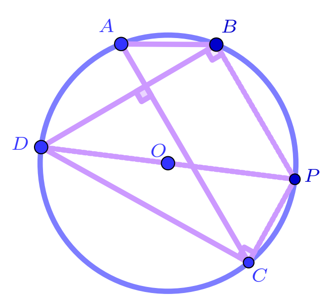
$$
\begin{aligned}
&2!\left[4\left(\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\binom{5}{2}+\binom{6}{2}+\binom{7}{2}\right)+2\cdot\binom{8}{2}\right] \\
=&2!\left(4\cdot\binom{8}{3}+2\cdot\binom{8}{2}\right) \\
=&2!(4\cdot 56+2\cdot 28) \\
=&560.
\end{aligned}
$$

(Note that by the hockey-stick identity, $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\binom{5}{2}+\binom{6}{2}+\binom{7}{2}=\binom{8}{3}$.)

An 8x8 chessboard has 64 squares in total, so the number of possible outcomes is $2!\cdot\binom{64}{2}=4032$. The number of outcomes in which the king is in check is $448+448+560=1456$. Hence, the probability that the king is in check is $1456/4032=\boxed{13/36}$.

10. Extend $\overline{DO}$ to some point P on circle O so that $\overline{DP}$ is a diameter. By Thale's Theorem, $\angle DBP$ is a right angle. Since both $\overline{AC}$ and $\overline{BP}$ are perpendicular to $\overline{DB}$, they are parallel. It follows that chords $\overline{AB}$ and $\overline{PC}$ are congruent, so $\overline{PC}$ has length x. Again, by Thale's Theorem, $\angle DCP$ is a right angle. Thus, $\triangle DCP$ has legs of length x and $3x$, and a hypotenuse of length 2. By the Pythagorean Theorem, we have

$$
\begin{aligned}
&9x^2+x^2=4, \\
&x^2=\frac{4}{10}=\frac{2}{5}, \text{ and} \\
&x=\frac{\sqrt{2}}{\sqrt{5}}=\boxed{\frac{\sqrt{10}}{5}}.
\end{aligned}
$$



11. We first notice that Dave has to be a machine, because, if his statement were true, there would be 3 mathematicians (Alice, Dave, and Emily).

Now, let's assume Alice is a mathematician. Since there can only be one more mathematician, Bob and Fred must both be machines. If one is a mathematician, the other must also be a mathematician. Emily's statement has to be false for Dave's statement to be false. That leaves Charlie to be the other mathematician, besides Alice. However, if Charlie's statement is true, then he is both a mathematician and the culprit. This is a contradiction because we know Emily's statement is false.

Therefore, Alice is a machine. Her statement is false, so she is the culprit. This also means Bob is a mathematician, and thus Fred is also a mathematician. Because there are already 2 mathematicians, Charlie and Emily must both be machines. Charlie's statement is indeed false because Alice ate the cookie. Emily's statement is also false: Alice, the culprit, is not a mathematician.

To summarize, Bob and Fred are mathematicians and Alice, Charlie, Dave, and Emily are machines. Additionally, Alice is the culprit.

12. **a)** Let categorized_points be a list of all the categorized points. For each point p in this list, we can store its color and x and y coordinates.

    Let new_point be the point we want to categorize. It also has an x and y coordinate, but we don't know the color yet.

    Let min_dist be running minimum distance to any of the categorized points, and let color be the color of the closest point.

    Set min_dist to infinity, so any real number is smaller than it.

    For each point p in categorized_points, do the following:

    (a) Get the x and y coordinates of p. Let this be p_x and p_y.
    (b) Get the x and y coordinates of new_point. Let these be np_x and np_y.
    (c) Calculate the distance between them, d, using the distance formula: $d = \sqrt{(p_x - np_x)^2 - (p_y - np_y)^2}$.

    If d is less than min_dist, set min_dist to d and set color to the color of p. Otherwise, do nothing.
    Set the color of new_point to the value of color.

    b) Instead of having a single min_dist variable, we could have 3 separate variables keeping track of the 3 smallest distances. We can let these be *dist1*, *dist2*, where *dist1* $<$ *dist2* $<$ *dist3* at all times. We can compare a new distance to all three of these to update them.

(a) If the new distance is smaller than all of them, set *dist3* to *dist2*, then set *dist2* to *dist1*, and then set *dist1* to the new distance.

(b) If the new distance is between *dist1* and *dist2*, set *dist3* to *dist2*, then set *dist2* to the new distance.

(c) If the new distance is between *dist2* and *dist3*, set *dist3* to the new distance.

We will also need 3 variables storing the colors of the 3 closest points, *color1*, *color2*, *color3*. After the loop in our pseudocode, we can check whether red appears at least twice in *color1*, *color2*, and *color3*. If so, the new point should be red; otherwise, it should be blue.